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On the Valuation of Reversionary Life Interests. By THOMAS BOND SPRAGUE, M.A., *Vice-President of the Institute of Actuaries.*

[Read before the Institute, 30th November, 1868.]

THE valuation of reversions, absolute and contingent, is a matter of great and growing importance. The greater part of the landed property in the United Kingdom being settled in strict entail, proposals are being constantly made by the expectant heirs for loans upon their reversionary estates; and so long as the present law and practice of settlement prevail, so long will the making advances on the security of reversionary interests in landed property continue to be an important part of the business of such Life Insurance Companies as are wise enough to cultivate it. Although the amount of the landed property in the country remains unaltered, yet the rentals are steadily increasing, and consequently the amount of the money which can be employed in the way above described may also be expected to increase. Again, the vast and rapid increase of wealth among the middle classes of the country has led to a great increase in the number of life interests and reversions under marriage settlements and wills; and in consequence of the many vicissitudes in trade, such reversionary interests in personal property are constantly offered for sale in increasing numbers. There is also, perhaps, a growing disposition, when the owners of reversionary interests in either real or personal estate become involved, to press their interests for public sale, instead of arranging privately a sale to some member of the family. On the other hand, as it is becoming better known how ready Offices of good standing are to make advances on contingent reversionary interests, applications are now made to them, which would formerly have been made to private money lenders, whose charges were, to say the least, excessive. From these combined causes, the number of reversions in the market is increasing, and seems likely to increase.

There are in existence several Reversionary Interest Societies founded for the purpose of employing the capital of the shareholders in the purchase of reversions; but these investments appear to be even better suited for Life Insurance Companies. The latter Companies, when once well established, have for a long series of years rapidly increasing assets, for which investments have continuously to be found. They can estimate within narrow limits the probable amount of the claims upon them; and they require to have only a small portion of their assets readily convertible. They can therefore afford to invest a large portion of their assets upon loans for long terms, such as mortgages of parochial rates and drainage rent-charges; and they thus obtain the advantage of the higher rate of interest which such securities produce, as compared with ordinary mortgages and the other securities for which there is a greater demand by trustees and private individuals. For the same reasons, they can afford to lock up considerable sums in the purchase of reversions, which on the average produce a still higher rate of interest than any of the securities above referred to. In the case of the purchase of an absolute reversion, it is a serious objection to an individual purchaser, that he receives no income from his investment; and in the case of a contingent reversion, there is the further objection that he has to incur a fresh outlay every year for payment of premiums; but so far from this being an objection when an Insurance Company is concerned, it is matter of congratulation when a contingent reversion, or reversionary life interest is purchased, that a remunerative investment has been made for part of the surplus assets of the future as well as of the present.

Before proceeding to the more particular object of the present paper, which is the explanation of some new formulæ for calculating the value of reversionary life interests, it will be convenient to consider the existing formulæ for the purchase of absolute and contingent reversions. It was formerly, I believe, the universal practice to value absolute reversions offered for sale by the ordinary tables of single premiums, using such a rate of interest,—5, 6, or 7 per cent—as circumstances seemed to call for. But in the second volume of this *Magazine*, page 162, Mr. Jellicoe, in the course of a remarkably able and comprehensive paper "On the contrivances required to render Contingent Reversionary Interests Marketable Securities," proposed a new method. His formula for the value of the reversion to £1 on the death of a person, aged x years, is $1-d(1+a_x)$, where d is the discount on £1 payable at the end of a year, and is to be taken at 5 per cent interest; and $1+a_x$ is the value of the annuity-due on the life, and is to be taken at $3\frac{1}{2}$ per cent. We may thus write the formula,

$$1-d_5(1+a_x)_{3\frac{1}{2}} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

How far this method has been adopted in practice by Actuaries, I am unable to say. It has been adopted by Mr. Tucker, among others, who speaks in praise of it in vol. v., pp. 166 and 242. (See also Mr. Day's paper on the Purchase of Life Assurance Policies as an Investment, vol. viii., page 326.) The only other discussion of the formula I can find is contained in Mr. Scratchley's Work on "Associations for Provident Investment." Mr. Jellicoe and Mr. Tucker urge the adoption of the formula on the ground that the value of a reversion should be such as will enable the purchaser to protect himself from all contingencies. It should be such as to allow of his purchasing an annuity that will furnish interest at five per cent on his total outlay until the reversion falls in. For it is not reasonable (Mr. Jellicoe in substance says) that the purchaser of an "isolated reversion" should be exposed to the risk of heavy loss in consequence of the tenant for life attaining extreme old age; and if he is secured against such loss by the purchase of the annuity above supposed, he can only fairly require such a rate of interest as is yielded by good mortgages. Mr. Scratchley, on the contrary, maintains that "the formula cannot be supported by any satisfactory reasoning, and is objectionable from its giving *negative* results for ages under a certain point." (See however the remarks in vol. vii., page 54.) Mr. Scratchley proceeds to recommend a new formula for the valuation of absolute reversions,

$$\frac{(\varpi_x)_5}{d_5+\varpi'_x} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

(where ϖ_x is the net premium calculated at 5 per cent, and ϖ'_x the office premium, for an insurance of £1 on the life of x); concerning which it may suffice to say that it does not seem to have been noticed, much less adopted, by any other actuary; and that the examples he gives of the values resulting from its use prove it to be altogether unsuitable for practical transactions.

Anything that is recommended by the high authority of Mr. Jellicoe and Mr. Tucker calls for the most careful consideration; and any actuary who dissents from their views should be prepared to state fully and clearly his reasons for such dissent. I will therefore do my best to explain the reasons which have induced me for a long time past to abandon altogether the use of Mr. Jellicoe's formula.

First, then, when the question arises,—*What is the market value of a certain reversion?*—I object to introducing the idea of the purchase of *isolated* reversions. There is in practice a very ready market for reversions; and indeed, whenever an absolute reversion free from all objection is offered for sale by public auction, there is a keen competition among the Reversionary Companies and other purchasers to obtain it. The market price then must certainly be that price which the principal purchasers,—the Reversionary Companies—find it worth their while to give; and the question resolves itself into determining upon what principle these Companies should regulate their purchases.

This brings me to my second objection to the new formula; viz., that it supposes an annuity to be purchased to provide interest on the outlay until the reversion falls in, whereas such an annuity is never actually purchased in practice. This being the case, if the value of a reversion is estimated by Mr. Jellicoe's formula, allowing 5 per cent interest to the purchaser, while the annuity is calculated at $3\frac{1}{2}$ per cent only, the purchaser of a large number of reversions, such as a Reversionary Society, would make a profit on the assumed grant of these annuities, in addition to the 5 per cent. It therefore appears to me much more satisfactory to include the whole profit of the transaction under one aspect, and to assume that the reversions are on the average bought on such terms as to yield a higher rate of interest. Assume, then, that a Reversionary Company buys its reversions at prices found from the six per cent single premiums, then if the number is sufficiently large, they will on the average fall in at such times as will return to the Society the cost together with six per cent compound interest. If the expenses of management are one per cent per annum on the capital, then the shareholders will receive five per cent on their capital. It cannot of course be supposed that the shareholders will be satisfied with a less return than this; and it would probably be too sanguine to assume that the expenses of these Companies can ever be permanently reduced much below one per cent per annum; and we may therefore fairly take six per cent as the proper rate upon which the market value of reversions may be estimated. There is however, I think, very little doubt that reversions dependent upon young lives may be bought to pay a higher rate of interest than those dependent upon old lives—say over 60; for this reason, that private purchasers who will often bid for the latter, avoid the former, on account of the smaller probability of their living to see the reversion fall in.

Such, then, being the terms on which the principal purchasers deal, the prices will in general be the same for other purchasers. I have said that these investments are even better suited for Insurance Companies, than for the Reversionary Companies. That this is so will appear when it is considered that an Insurance Company possessed of large funds may transact the whole business of a Reversionary Company without any increase of expense, beyond, perhaps, the employment of an additional clerk. It will not perhaps be too much to prophesy that as the Insurance Companies become more alive to the advantages of this class of business, they will drive the Reversionary Companies out of the field. In this connection it may be well to advert to an objection often raised against the purchase of reversions, viz., that it encourages young men to anticipate their future expectations. That this objection has no real weight will appear from the undoubted fact that reversions are almost always sold to pay debts already incurred; that the spending,—the real anticipation of the future,—has already taken place before the reversion is offered for sale.

I will next proceed to consider the value of contingent reversions. Mr. Jellicoe's formula for the value of a contingent reversion payable on the death of y if x be then alive, is $1-(d_5+\mathrm{P})(1+a_{xy})_{3\frac{1}{2}}$, where P is the office premium for the insurance of £1 on the life of x against that of y, and a_{xy} is to be taken at $3\frac{1}{2}$ per cent interest. This formula is based on the supposition that the amount of the reversion being £1, an insurance of £1 is effected on the life of x against y, the annual premium for which is P; so that, if x die first, £1 is received at the end of the year under the policy; and if y die first, the reversion becomes payable. In this way the reversion is virtually rendered absolute, and is payable on the failure of the joint lives of x and y;

or rather at the end of the year in which that failure takes place. The value of such absolute reversion would be by Mr. Jellicoe's formula

$$1-d_5(1+a_{xy})_{3\frac{1}{2}}.$$

But from this must be deducted the value of the premiums on the insurance, which the purchaser will have to pay. Mr. Jellicoe estimates this at such a sum as an annuity office would require for the grant of an annuity of £P, *i.e.*

$$\text{P}(1+a_{xy})_{3\frac{1}{2}}.$$

Thus the value of the contingent reversion becomes

$$1-(d_5+\text{P})(1+a_{xy})_{3\frac{1}{2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Assuming the purchaser in this case (or in the case already considered of an absolute reversion) actually to purchase the annuities supposed, his total outlay will be $1-d$, or v. This is easily verified. Thus the annual interest on v is d; the first premium is P, and the cost of an annuity to pay both interest and premium, *i.e.* $(d+\text{P})$, at the end of each year during the joint lives, is $(d+\text{P})a_{xy}$.

Thus, purchase money $=1-(d+\text{P})(1+a_{xy})$

first premium $=\text{P}$

cost of annuity of $(d+\text{P})=(d+\text{P})a_{xy}$

and the sum of these is $1-d$, as stated above, or v. In this way, then, the total present cost to a purchaser equals the amount of the reversion, less only a year's discount.

In lieu of the above formula, (3), I propose to substitute the following,

$$1-(d_6+\text{P})(1+a_{xy})_6 \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

which differs from it, in supposing the discount and the annuity during the joint lives to be both computed at six per cent interest; P having the same meaning as before, viz. the annual office premium for an insurance of £1 on the life of x against y.

Assuming as before the policy of £1 to be effected, and the reversion to be thus rendered absolute, its value, according to the views I have enforced above, will be A_{xy}, or

$$1-d_6(1+a_{xy})_6.$$

The question then arises, what allowance is to be made, or what deduction is to be made from the above value, on account of the cost of the premiums during the joint lives? The deduction made by Mr. Jellicoe's process, *i.e.*, $\text{P}(1+a_{xy})_{3\frac{1}{2}}$ would be the proper one, if an annuity were actually purchased to provide for the premiums as they fall due, or if the premiums were paid up in full at once, or, lastly, if the purchaser found it desirable to keep a sum of money invested in Consols or other readily convertible security, in order to provide for payment of the premiums by means of the interest and sale of portion of the principal from time to time. But the principal purchasers of reversions—the Reversionary and Insurance Companies—do not come under any one of these heads. They would not think of buying an annuity to provide for the premium; nor would they, I imagine, effect the policy by way of single premium. Nor, lastly, is it in the least necessary for them to keep a fund especially appropriated to payment of premiums. On the contrary, they have income coming in from various sources, which they can rely upon with confidence to furnish the funds for payment of the premiums as they fall due. It appears to me therefore that the value of the annuity, or the cost of the premiums, should be calculated at the same rate of interest—say, six per cent—as is used in calculating the value of the reversion. The deduction will be therefore, $P(1+a_{xy})_6$; and we thus get the formula (4) given above. If a sufficient number of contingent reversions be bought at prices found from this formula, they will on the average fall in at such a rate as to return the total outlay in purchase money and premiums, with compound interest at six per cent.

The only valid objection to the above reasoning I am aware of is the one adverted to by Mr. Tucker at the beginning of his paper, vol. v., page 239. The amounts of the reversions purchased being different, some perhaps much larger than the others, it may be said that the principles of average do not properly apply. This appears to me, however, only an argument against purchasing any reversion that shall greatly exceed all the others. It may also, perhaps, be properly admitted as an argument for valuing very large reversions by a less liberal standard than would be applied in other cases. Indeed, the principle is recognized in many instances that any transaction of an unusual character must be charged with a larger margin for contingencies. This is conspicuously the case with "insurances against issue," the premiums for which are invariably calculated with a larger loading than is required in life insurances.

The principles I have above laid down may be applied to find the value of an ordinary policy of insurance, considered as an investment. The sum assured being £1, and the annual premium

P, the value of the policy, supposing the premium to be due and unpaid, will be,

$$(A_x)_6 - P(1+a_x)_6,$$

or

$$1-(d_6+P)(1+a_x)_6 \quad . \quad . \quad . \quad . \quad (5)$$

Having thus cleared the ground, we are now in a position to consider the still more important practical question of the value of a reversionary life interest.

Mr. Jellicoe's formula for the value of a reversionary annuity of £1 for the life of x after the death of y, (which I believe has been very generally, or almost universally, employed,) is

$$\frac{1}{P_x+d_5} - 1 - (a_{xy})_{3\frac{1}{2}},$$

where P_x is the office premium for the insurance of £1 on the life of x, d_5 is the discount on £1 for a year at 5 per cent, and a_{xy} is the annuity on the joint lives calculated at $3\frac{1}{2}$ (or perhaps $3\frac{1}{4}$ or 3) per cent.

The object of this formula is very easily explained. If the annuity of £1 on the life of x were in possession, its value would be, according to the usual well known formula, $\frac{1}{P_x+d_5} - 1$. But the annuity under consideration does not commence until the death of y, if x be then alive, and requires the addition of an annuity of £1 during the joint lives, to convert it into an immediate annuity for the life of x. Now such an annuity could be purchased from a well established Insurance or Annuity Company for $(a_{xy})_{3\frac{1}{2}}$; and deducting this from the value of the annuity in possession, the value of the annuity in reversion is, as given above,

$$\frac{1}{P_x+d_5} - 1 - (a_{xy})_{3\frac{1}{2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

With regard to this formula, it is first to be observed, that the reversionary annuity runs in practice from the death of y; whereas it is virtually supposed in the formula that it runs from the end of the preceding year. For the tabular annuity, a_{xy}, is the value of an annuity which ceases at the end of the year before that in which the joint existence of the two lives x and y fails. By the above formula (6) therefore the purchaser of the reversionary annuity is supposed to receive on the average half a year's annuity in the event of y dying before x, which he will not receive in practice. In strictness, then, there should be subtracted from the above formula $\frac{1}{2}A_{\overset{1}{x}y}$.

It may, perhaps, be considered that in most cases a_{xy} will practically purchase an annuity payable up to the day of the failure of the joint lives (or a *complete annuity*); but this will certainly not be the case when y is very old; and, in that case, the above formula will give too large a value to the reversionary annuity.

I have next to repeat the remark I have made in considering the cases of absolute and contingent reversions, that an Assurance or Reversionary Company purchasing a reversionary annuity would not in practice ever think of purchasing also an annuity during the joint lives; but would take upon itself the risk of loss in consequence of the joint duration of the lives being unusually extended. The question then is, what allowance is to be made for that risk? I would urge that instead of using the two rates of interest $3\frac{1}{2}$ and 5 per cent, the real bearing of the transaction as regards the purchaser, will he better seen, if we employ the same rate throughout. In other words, let the deduction for the value of the annuity during the joint lives be calculated at the same rate of interest as the annuity in possession would yield. In that case, it would not be sufficiently remunerative to assume 5 per cent as the basis of the calculation; and it will be better to take 6 per cent,—the rate used in estimating the values of absolute and contingent reversions. Then the formula becomes, adopting the correction pointed out above,

$$\frac{1}{P_x+d_6}-1-(a_{xy})_6-\tfrac{1}{2}(A_{\underset{xy}{1}})_6 \quad . \quad . \quad . \quad (7)$$

The practical effect of using this formula will be to give a larger value to the reversionary life interest when the reversion is remote, the tenant for life being comparatively young, and to give a smaller value when the tenant for life is very old.

Mr. Scratchley has given (Appendix on Post Obits, p. 18) a formula for the value of a reversionary life interest which becomes when transformed into the notation used in this paper

$$\frac{A_{xy}}{P_x+d}-(1+a_{xy})\frac{P_x}{P_x+d}. \quad . \quad . \quad . \quad . \quad (8)$$

We are told that A_{xy}, a_{xy}, and d are to be "taken at a high rate of interest varying from 6 to 8 per cent"; but these directions are so vague that the formula can scarcely be considered a practical one. The above formula reduces at once to

$$\frac{1}{P_x+d}-1-a_{xy},$$

and Mr. Scratchley has not noticed the necessity of the correction given by the last term of (7).

In using the above formula (7) it is assumed that on the average six per cent interest is realized by the purchaser on the transaction throughout its currency, until the younger life dies; and there can be little doubt, I think, that it is very well worth while for an Insurance or a Reversionary Society to purchase reversionary annuities on such terms. But I am further of opinion that it would be advantageous to an Insurance Office to purchase these reversionary annuities upon terms somewhat higher still. Assuming that such an office would grant loans on life interests at 5 per cent, or would purchase an immediate annuity, perfectly well secured, to pay 5 per cent interest; and would purchase an absolute or contingent reversion to pay six per cent interest; it should seem that the office ought to be satisfied if it purchases reversionary life interests at such prices as will on the average return six per cent interest until the reversions fall in, and five per cent afterwards. I consider it, however, doubtful whether it would be worth while for a Reversionary Society to purchase reversionary life interests on these terms. The advantage of the purchase to an Insurance Company consists not only in the high rate of interest, but in the large insurances which the transaction introduces. If the insurance does not exceed the amount which the office usually retains at its own risk, the office will have the whole of the profit on the insurance; and if the insurance is larger, and has to be distributed among several offices, then the office is indirectly benefited by receiving from those offices other policies in return. For this reason, I consider that an Insurance Company may afford to enter on these transactions upon terms which shew a smaller profit than would satisfy a Reversionary Company.

In order to obtain a formula for the value of a reversionary life interest on the above suppositions, we proceed as follows:—

The value of an annuity of £1 in possession for the life of x is $\dfrac{1}{P_x+d_5}-1$, and the amount of the insurance to be effected in connection therewith will be $\dfrac{1}{P_x+d_5}$; the annual premium thereon being $\dfrac{P_x}{P_x+d_5}$.

Now, supposing this insurance to be effected, the value of the life interest and the insurance together, remains the same what-ever the number of complete years which may elapse. Whenever therefore the reversionary life interest may fall in, the value of the annuity and the insurance together will *at the end of the year in which y dies* be (the premium on the policy being supposed due and unpaid)

$$\frac{1}{P_x + d_5} - 1\,;$$

or if we assume that a year's annuity is payable at the end of the year in which y dies, the value will be

$$\frac{1}{P_x + d_5}.$$

The *present value* of the annuity and the insurance together at six per cent interest will therefore be

$$\frac{(A_{xy})_6}{P_x + d_5}.$$

But since the reversionary annuity in practice runs only from the death of y, there is on the average only half a year's annuity due at the end of the year in which y dies; and the present value becomes therefore,

$$\frac{(A_{xy})_6}{P_x + d_5} - \tfrac{1}{2}(A_{\underset{xy}{-1}})_6.$$

Again, since the purchaser has to pay the premiums on the insurance during the joint lives, the present value of such premiums must also be subtracted, which, still using six per cent interest, will be

$$(1 + a_{xy})_6 \times \frac{P_x}{P_x + d_5}.$$

Thus we get as the consideration to be paid for the reversionary annuity of £1

$$\frac{(A_{xy})_6}{P_x + d_5} - \tfrac{1}{2}(A_{\underset{xy}{-1}})_6 - (1 + a_{xy})_6 \times \frac{P_x}{P_x + d_5}$$

which becomes, since $A_{xy} = 1 - d(1 + a_{xy})$,

$$\frac{1}{P_x + d_5} - \frac{P_x + d_6}{P_x + d_5}(1 + a_{xy})_6 - \tfrac{1}{2}(A_{\underset{xy}{-1}})_6 \quad . \quad . \quad . \quad . \quad (9)$$

Or, since y is usually much older than x, and $A_{\underset{xy}{-1}}$ is therefore not much less than A_{xy}, we have the simpler formula approximately true

$$\left(\frac{1}{P_x + d_5} - \tfrac{1}{2}\right)(A_{xy})_6 - (1 + a_{xy})_6 \times \frac{P_x}{P_x + d_5}$$

which reduces to

$$\frac{1}{P_x+d_5}-\tfrac{1}{2}-(1+a_{xy})_6\times\left\{\frac{P_x+d_6}{P_x+d_5}-\frac{d_6}{2}\right\}\quad . \quad . \quad . \quad (10)$$

It will be noticed that the substitution of A_{xy} for $A_{\overset{1}{xy}}$ is in favour of the purchaser.

This, then, is the practical formula which I propose should be used in lieu of Mr. Jellicoe's, for calculating the value of reversionary life interests. It will be found to bring out a considerably larger value for such interests when the tenant for life is under 60, but a somewhat smaller value when the tenant for life is more than 65.

I submit that the above formula shows more accurately than the ordinary one the real working of the transaction as it affects the office making an advance. It is true that if an annuity were actually purchased for the joint lives, its cost would have to be calculated at from 3 to $3\frac{1}{2}$ per cent. But, in practice, no such annuity is ever purchased. The office making the advance pays a sum to the borrower, and pays every year the premiums on the insurances effected, *i.e.* virtually makes further advances from year to year. Consistently with what I have said above as to the valuation of absolute and contingent reversions, I hold that the same rate of interest ought to be used in calculating the values of A_{xy} and a_{xy} in the two terms

$$A_{xy}\left(\frac{1}{P_x+d_5}-\tfrac{1}{2}\right),\quad (1+a_{xy})\frac{P_x}{P_x+d_5};$$

and assuming that they are both calculated at six per cent interest, I say that if the original advance and the premiums subsequently paid are accumulated at compound interest at six per cent, then on the average their amount at the time the reversion falls in will be equal to the then value, calculated at five per cent interest, of the annuity and the policies.

The value of the reversionary life interest found by means of the formula (10) being greater than that given by Mr. Jellicoe's formula, when the tenant for life is under 60, the amount of the reversionary annuity and of the redemption money for a given advance will be less; and in the case of remote reversionary interests, the use of the formula (10) will generally give a positive value and enable an advance to be carried out when it would be impracticable according to the ordinary formula. But these, it will be seen, are the cases in which the amount of the insurance bears the largest proportion to the advance; and in these cases an Insurance Company making the advance would directly or indirectly obtain a large profit on the insurance, and may therefore fairly be satisfied with a rather less profit in other respects.

Whichever of the formulæ is used, the amount of the policy to be effected to secure the return of the capital, and the redemption money, or sum for which the annuity may be redeemed after the reversion has fallen in, will be the same as if the annuity were in possession; *i.e.*, the annuity being £1, the amount of the policy, when either of the formulæ (6) or (10) is used, will be $\frac{1}{P_x+d_5}$ and the redemption money $\frac{v_5}{P_x+d_5}$, or $\frac{1-d_5}{P_x+d_5}$. And when formula (7) is used, the policy will be $\frac{1}{P_x+d_6}$, and the redemption money, $\frac{v_6}{P_x+d_6}$. As Mr. Jellicoe has pointed out, upon the annuity being redeemed, equity requires that the policies effected by the purchaser should be assigned to the vendor; and I believe that the rule of law alluded to by Mr. Jellicoe, that in the absence of any express stipulation, the grantor of the annuity shall not be entitled to the policy, has now been overruled by a decision of Vice-Chancellor Stuart. It is usual to stipulate that upon the redemption of the annuity a proportionate part of it shall be paid from the death of the tenant for life or from the date of the last payment; but here again equity is satisfied if instead of a proportion of the annuity, interest is paid on the redemption money at the agreed rate, five or six per cent.

Instead of insuring the full sum given by the above formula, it will generally be better, and more particularly in the case of remote reversions, to insure *with profits* for a considerably smaller sum, trusting to the reversionary bonuses to increase the insurance in process of time to the required amount. For the bonuses are generally far more than an equivalent for the difference between the participating and non-participating premiums; and if the bonuses are added to the sum assured, we shall virtually have an increasing insurance, amounting at last to a far larger sum than could have been insured for the same premium on the non-participating scale. And as the amount of the advance is increasing year by year, through the payment of premiums and the operation of compound interest, such increasing insurance is better adapted to the circumstances of the case than a uniform insurance would be.

If a reversionary life interest be purchased in its entirety, the policy should be made "whole world"; as otherwise the purchaser will be exposed to the risk of having to pay a heavy additional premium in the event of the life proceeding to an unhealthy climate. If the extra premium for the whole world licence be an annual one (as is most commonly the case, the charge ranging in ordinary cases from 2*s.* 6*d.* to 10*s.* per £100 assured), P_x will have to be increased in calculating the value of the life interest. If, however, it should be a single premium (say, 10*s.* or £1 per £100 assured) this single payment must be deducted from the value of the life interest, as found from the formula.

But if a portion only of the reversionary life interest be purchased, as in the common case when an advance is made by way of reversionary annuity on security of a reversionary life interest, and there is a considerable margin left, it is not necessary that the policy should be whole world; for in that case it may be stipulated that for every £100 that the purchaser has to pay in extra premiums, the amount of the reversionary annuity shall receive a fixed increase.

It may be useful to give in conclusion the formulæ for the amounts of the reversionary annuity and the redemption money, in consideration of a present advance of £1; although in practice it will be found more convenient to deduce these from the value of the reversionary annuity as found by the formula (6) (7) or (10). Taking the first of these, we see that a reversionary annuity of £1 is worth $\frac{1}{P_x+d}-1-a_{xy}$, and may be redeemed at any time for $\frac{1-d}{P_x+d}$. Hence a reversionary annuity of $\frac{P_x+d}{1-(1+a_{xy})(P_x+d)}$ is worth £1, and is redeemable for $\frac{1-d}{1-(1+a_{xy})(P_x+d)}$.

Or, the advance being £1·

$$\text{the reversionary annuity} = \frac{P+d_5}{1-(1+a_{xy})_{3\frac{1}{2}}\cdot(P+d_5)}$$

$$\text{the amount of the policy} = \frac{1}{1-(1+a_{xy})_{3\frac{1}{2}}\cdot(P+d_5)}$$

$$\text{and the redemption money} = \frac{1-d_5}{1-(1+a_{xy})_{3\frac{1}{2}}\cdot(P+d_5)}$$

where P is the premium for insuring £1 on the life of x.

Next, take the formula (7), but substitute for $A_{\overset{1}{xy}}$, which is troublesome to calculate, A_{xy}, which is nearly equal to it in the common case of y much older than x; then, remembering that $A_{xy}=1-d(1+a_{xy})$, we get the working formula for the value of a reversionary annuity

$$\frac{1}{P+d_6}-\tfrac{1}{2}-\left(1-\frac{d_6}{2}\right)(1+a_{xy})_6 \quad . \quad . \quad . \quad . \quad (11)$$

The advance being, as before, £1, we have now

the reversionary annuity

$$=\frac{P+d_6}{1-\dfrac{P+d_6}{2}-(1+a_{xy})_6\cdot\left(1-\dfrac{d_6}{2}\right)(P+d_6)}$$

the amount of the policy

$$=\frac{1}{1-\dfrac{P+d_6}{2}-(1+a_{xy})_6\cdot\left(1-\dfrac{d_6}{2}\right)(P+d_6)}$$

and the redemption money

$$=\frac{1-d_6}{1-\dfrac{P+d_6}{2}-(1+a_{xy})_6\cdot\left(1-\dfrac{d_6}{2}\right)(P+d_6)}.$$

Lastly, taking the formula (10),

the reversionary annuity

$$=\frac{1}{\dfrac{1}{P+d_5}-\tfrac{1}{2}-(1+a_{xy})_6\cdot\left(\dfrac{P+d_6}{P+d_5}-\dfrac{d_6}{2}\right)}$$

$$=\frac{P+d_5}{1-\dfrac{P+d_5}{2}-(1+a_{xy})_6\cdot\left\{P+d_6-\dfrac{d_6}{2}(P+d_5)\right\}}$$

the amount of the policy

$$=\frac{1}{1-\dfrac{P+d_5}{2}-(1+a_{xy})_6\cdot\left\{P+d_6-\dfrac{d_6}{2}(P+d_5)\right\}}$$

and the redemption money

$$=\frac{1-d_5}{1-\dfrac{P+d_5}{2}-(1+a_{xy})_6\cdot\left\{P+d_6-\dfrac{d_6}{2}(P+d_5)\right\}}.$$

By the help of these formulæ, it is not difficult to prove that the amount of the policy is always greater when formula (11) is used, than that given by the use of (10). But that the redemption money is not always greater. In fact, the redemption money obtained from (11) is greater or less than that obtained from (10) according as $1+a_{xy}$ is greater or less than $\dfrac{1}{\dfrac{2P}{1-P}+d_6}$. The values however of the redemption money as found from these two formulæ are very nearly equal, as will be seen by inspection of the following tables.

TABLE I.—*Immediate Annuities,* 5 *and* 6 *per Cent.*

Table showing the amounts of the policy and of the redemption money when an annuity of £1 *is purchased,—allowing for insurance at the average annual premiums here given, and returning the purchaser* 5 *and* 6 *per Cent respectively on his outlay.*

Age.	Premium.	5 PER CENT.		6 PER CENT.	
		Policy.	Redemption Money.	Policy.	Redemption Money.
x	P_x	$\frac{1}{P_x+d_5}$	$\frac{1-d_5}{P_x+d_5}$	$\frac{1}{P_x+d_6}$	$\frac{1-d_6}{P_x+d_6}$
	£ s. d.				
20	1 15 1	15·347	14·616	13·488	12·724
25	1 19 6	14·843	14·136	13·098	12·356
30	2 4 9	14·286	13·606	12·661	11·945
35	2 11 2	13·661	13·010	12·168	11·479
40	2 19 2	12·953	12·336	11·604	10·947
45	3 5 7	12·134	11·556	10·942	10·323
50	4 3 6	11·189	10·656	10·168	9·592
55	5 2 0	10·140	9·657	9·294	8·768
60	6 7 8	8·973	8·546	8·304	7·834

TABLE II.—*Values of Reversionary Annuities.*

(*a*)

As found by Mr. Jellicoe's formula No. (6), $\frac{1}{P+d_5}-(1+a_{xy})_{3\frac{1}{2}}$.

Younger Age. x.	Difference of age $=y-x$. 10	15	20	25	30	35	40	45	50
20	− 1·308	− ·689	·098	·907	1·990	3·422	4·946	6·221	7·753
25	− ·864	− ·129	·633	1·672	3·061	4·549	5·795	7·301	8·709
30	− ·350	·354	1·338	2·677	4·123	5·338	6·814	8·201	9·204
35	·014	·929	2·208	3·610	4·792	6·241	7·608	8·598	
40	·549	1·742	3·075	4·211	5·622	6·962	7·935		
45	1·156	2·404	3·480	4·849	6·167	7·130			
50	1·767	2·739	4·023	5·286	6·216				
55	2·137	3·278	4·436	5·297					
60	2·547	3·574	4·347						

(*b*)

As found by formula (11), $\frac{1}{P+d_6}-\frac{1}{2}-\left(1-\frac{d_6}{2}\right)(1+a_{xy})_6$.

Younger Age. x.	Difference of age $=y-x$. 10	15	20	25	30	35	40	45	50
20	·700	1·005	1·422	1·840	2·462	3·377	4·416	5·294	6·433
25	·803	1·197	1·595	2·193	3·085	4·101	4·961	6·082	7·190
30	·961	1·334	1·906	2·769	3·761	4·600	5·701	6·793	7·597
35	1·010	1·549	2·380	3·345	4·166	5·246	6·326	7·119	
40	1·196	1·981	2·907	3·698	4·753	5·812	6·593		
45	1·461	2·339	3·094	4·124	5·169	5·942			
50	1·771	2·461	3·435	4·440	5·191				
55	1·921	2·795	3·725	4·422					
60	2·153	2·985	3·621						

TABLE II.—(continued).

(*c*)

As found by the new formula No. (10), $\frac{1}{P+d_5} - \frac{1}{2} - (1+a_{xy})_6 \times \left(\frac{P+d_6}{P+d_5} - \frac{d_6}{2}\right)$

Younger Ages. x	Difference of age $=y-x$. 10	15	20	25	30	35	40	45	50
20	·816	1·165	1·640	2·118	2·828	3·873	5·059	6·062	7·362
25	·930	1·378	1·831	2·511	3·525	4·681	5·659	6·933	8·194
30	1·107	1·530	2·177	3·153	4·276	5·227	6·473	7·710	8·619
35	1·157	1·764	2·700	3·787	4·711	5·928	7·144	8·038	
40	1·359	2·238	3·275	4·160	5·342	6·528	7·402		
45	1·646	2·623	3·462	4·608	5·770	6·630			
50	1·976	2·737	3·811	4·921	5·750				
55	2·123	3·078	4·095	4·858					
60	2·353	3·255	3·938						

TABLE III.—*Annuity which £1 will purchase, and its Redemption Money.*

Ages. x	y	J, (6). Annuity.	J, (6). Redemption Money.	6 PER CENT, (11). Annuity.	6 PER CENT, (11). Redemption Money.	S, (10). Annuity.	S, (10). Redemption Money.
Difference of age, 10 years.							
20	30	..	..	1·4286	18·178	1·2255	17·912
25	35	..	..	1·2453	15·387	1·0753	15·200
30	40	..	..	1·0406	12·429	·9033	12·290
35	45	71·4286	929·286	·9901	11·366	·8643	11·245
40	50	1·8215	22·470	·8361	9·153	·7358	9·077
45	55	·8651	9·997	·6845	7·066	·6075	7·020
50	60	·5659	6·030	·5647	5·416	·5061	5·393
55	65	·4679	4·519	·5206	4·564	·4710	4·549
60	70	·3926	3·355	·4645	3·639	·4250	3·632
Difference of age, 15 years.							
20	35	..	..	·9950	12·661	·8584	12·546
25	40	..	..	·8354	10·322	·7257	10·259
30	45	2·8249	38·435	·7496	8·954	·6536	8·893
35	50	1·0764	14·004	·6456	7·411	·5669	7·376
40	55	·5741	7·082	·5048	5·526	·4468	5·512
45	60	·4160	4·807	·4275	4·413	·3812	4·405
50	65	·3651	3·891	·4063	3·897	·3654	3·894
55	70	·3051	2·946	·3578	3·137	·3249	3·138
60	75	·2798	2·391	·3350	2·624	·3072	2·625
Difference of age, 20 years.							
20	40	10·2041	149·142	·7032	8·948	·6098	8·913
25	45	1·5798	22·331	·6270	7·747	·5461	7·720
30	50	·7474	10·169	·5247	6·267	·4593	6·249
35	55	·4529	5·892	·4202	4·824	·3704	4·819
40	60	·3252	4·012	·3440	3·766	·3053	3·766
45	65	·2874	3·321	·3232	3·336	·2889	3·339
50	70	·2486	2·649	·2911	2·792	·2624	2·796
55	75	·2254	2·177	·2685	2·354	·2442	2·358
60	80	·2300	1·966	·2762	2·164	·2539	2·170

TABLE III. (continued).

Ages.		J, (6).		6 per Cent, (11).		S, (10).	
x	y	Annuity.	Redemption Money.	Annuity.	Redemption Money.	Annuity.	Redemption Money.
Difference of age, 25 years.							
20	45	1·1025	16·114	·5435	6·916	·4721	6·900
25	50	·5981	8·455	·4560	5·634	·3982	5·629
30	55	·3736	5·083	·3611	4·313	·3172	4·316
35	60	·2770	3·604	·2990	3·432	·2641	3·436
40	65	·2375	2·930	·2704	2·960	·2404	2·966
45	70	·2062	2·383	·2425	2·503	·2170	2·508
50	75	·1892	2·016	·2252	2·160	·2032	2·165
55	80	·1888	1·823	·2261	1·982	·2058	1·988
Difference of age, 30 years.							
20	50	·5025	7·345	·4062	5·169	·3536	5·168
25	55	·3267	4·618	·3241	4·005	·2837	4·010
30	60	·2425	3·299	·2659	3·176	·2339	3·182
35	65	·2087	2·715	·2400	2·755	·2123	2·762
40	70	·1779	2·195	·2104	2·303	·1872	2·309
45	75	·1622	1·874	·1935	1·997	·1733	2·003
50	80	·1609	1·715	·1926	1·848	·1739	1·853
Difference of age, 35 years.							
20	55	·2922	4·271	·2961	3·768	·2582	3·774
25	60	·2198	3·107	·2438	3·012	·2136	3·020
30	65	·1873	2·548	·2174	2·597	·1913	2·603
35	70	·1602	2·084	·1906	2·188	·1687	2·195
40	75	·1436	1·732	·1721	1·884	·1532	1·890
45	80	·1403	1·621	·1683	1·737	·1508	1·743
Difference of age, 40 years.							
20	60	·2022	2·955	·2264	2·881	·1977	2·890
25	65	·1726	2·440	·2016	2·491	·1767	2·498
30	70	·1468	1·997	·1754	2·095	·1545	2·102
35	75	·1314	1·710	·1581	1·815	·1400	1·822
40	80	·1260	1·554	·1517	1·661	·1351	1·667
Difference of age, 45 years.							
20	65	·1607	2·349	·1889	2·404	·1650	2·412
25	70	·1370	1·937	·1644	2·031	·1442	2·039
30	75	·1219	1·659	·1472	1·758	·1297	1·765
35	80	·1163	1·513	·1405	1·613	·1244	1·619
Difference of age, 50 years.							
20	70	·1290	1·886	·1554	1·977	·1358	1·985
25	75	·1148	1·623	·1391	1·719	·1220	1·725
30	80	·1086	1·478	·1316	1·572	·1160	1·578

These tables have been calculated by Mr. Berridge, of the London and Provincial Law Assurance Society, and are arranged so as to show side by side the results of the principal formulæ mentioned in this paper. The premiums involved are the average of the non-participating premiums charged by six leading offices; they are given in Table I. This Table also shows the sum to be assured to cover the risk incident on the purchase of an annuity of £1, whether immediate or not; and the redemption money for the same; being the values of the expressions $\frac{1}{P_x+d}$ and $\frac{1-d}{P_x+d}$. They are calculated at 5 and 6 per cent.

The second table gives the value of a reversionary annuity of £1 according to three different methods of valuation. The first division of the table marked (a) results from the employment of Mr. Jellicoe's formula $\frac{1}{P_x+d_5}-1-(a_{xy})_{3\frac{1}{2}}$, the Carlisle $3\frac{1}{2}$ per cent annuity plus unity being subtracted from the values in the column of Table I. headed $\frac{1}{P_x+d_5}$.

In calculating the second division (b), formula (7) was used; the joint life single premium A_{xy} being substituted for the contingent premium $A_{\overset{1}{xy}}$. Interest at 6 per cent is therefore reserved throughout. Identical results would of course have been given by formula (11).

The third division (c) contains the values of a reversionary annuity of £1 according to my new formula (10), the Carlisle 6 per cent annuity being used in this and the previous division.

The third table gives the reversionary annuity which £1 will purchase and its redemption money, according to the same three methods, the values in the annuity columns being the reciprocals of those in Table II. The amounts of redemption money opposite to them result from their multiplication by the corresponding amounts in Tàble I., the five per cent values being employed for the columns headed J and S, and the six per cent values for the middle column; or rather the logarithms of which the first table contains the natural numbers. Where blanks occur the formula gives a negative value to the reversionary annuity.

The following summary of the discussion which followed the reading of the paper is abridged from the *Insurance Record.*

The PRESIDENT, in inviting discussion, while regretting the absence of Messrs. Jellicoe and Tucker, hoped that those gentlemen who were connected with Reversionary Societies would favour the meeting with their views. He thought that Mr. Sprague's proposed method of dealing with reversionary securities was one which, if acted on, would drive both the public and the Reversionary Societies out of the market.

Mr. BUNYON thought that Mr. Sprague was taking too sanguine a view of the purchase of reversions. Looking at Mr. Jellicoe's formula, he did not see what provision was made for "profit" on the transaction. If the profit was supposed to come from the rate of interest secured (say 5 per cent), it amounted to nothing more than a rate of interest, which they were able to obtain from certain other securities. If from the premium charged, a considerable portion did not reach them, by reason of their having to distribute the assurances, often in the vain hope of some day getting a return. If from the value of the annuity reserved at $3\frac{1}{2}$ per cent, he was still more doubtful, and was quite willing to let all annuity business on those terms go elsewhere. He was aware that considerable profit had been made upon reversionary transactions, and that one reason for granting better terms might be the fact that there was no selection against the purchaser. He thought, however, that as a rule there was no great value in selection, and concluded by entreating purchasers not to spoil the market.

Mr. H. AMBROSE SMITH preferred as a matter of account to abide by the principle typified in Mr. Jellicoe's formula, and in buying a reversion would also suppose the annuity bought, and set up the transaction in both the Investment Ledger and Life Contingency accounts. He thought Mr. Sprague's formula inapplicable where only a small number of contingencies were at stake.

Mr. HODGE felt inclined to agree with the mathematical views taken by Mr. Sprague, though in some points not so entirely. The market for reversionary securities was a limited one, and the competition of Assurance Companies would have the effect of raising the price of reversions, and might render it impossible for the Reversionary Companies to continue their business; a result not greatly to be lamented, as they would be enabled to dispose of their property at a considerable profit. He considered that the expectations of profit from these transactions were exaggerated. Of the seven Reversionary Companies which had started, three had ceased to exist; and of those surviving, none had their shares at a premium. The purchase of a contingent reversion, according to the ordinary formulæ laid down in the text books, cannot be said to be an investment, but a speculation. Such a transaction amounts to doing the business of an insurance office, without getting the profit. He believed that, in the cases which generally arose in practice, Mr. Jellicoe's formulæ for the valuation of contingent reversions would be inapplicable, and that even Mr. Sprague's would often bring out results so small that Reversionary Companies would decline to enter into the business. The latter formula [*i.e.* No. (4) in the paper] had been known to many actuaries and commonly used for many years. He was not aware of any case in which a Reversionary Company had acted upon Mr. Jellicoe's formula; but it is one properly adapted for private purchasers, who can thus obtain a certain result without taking upon themselves the chance of either great loss or great gain. The method usually adopted by purchasers of covering themselves by assuring the life of the reversioner against that of the life-tenant is convenient and advantageous to the vendor, but it is not so strictly legal as the insurance of a sum payable upon the decease of the life-tenant, contingent upon his dying second. He thought that all the formulæ, though convenient, did not quite meet the actual conditions of practice, for annuities were generally payable either half-yearly or quarterly, and the life-tenant of an estate was entitled to an apportionment of the income up to the date of his death. He should be glad to hear of some method for approximately arriving at the value of contingent reversions, including the benefit of survivorship.

Mr. Bailey congratulated the Institute that a paper of so practical a character was presented on the opening night of the session. He did not agree with Mr. Sprague's reasons for adopting the rates of interest employed in the paper. He thought that at present reversions should be valued at 6 per cent, because in fact that was the price which at present ruled in the market; but, for the future, owing to the protection afforded by the recent legislation, their value would no doubt be enhanced, and the rate of interest consequently obtainable be lessened. Assurance Companies might with advantage purchase at a lower rate than 6 per cent, because they cannot realize such a return upon their ordinary investments. The principle originally proposed by Mr. Griffith Davies for valuing an isolated annuity was extended by Mr. Jellicoe to the case of a single reversion. He (Mr. Bailey) did not, however, think that it had been much adopted in practice; for, the capital of the purchaser being secure, his sole risk was as to the rate of interest which he would ultimately make, and he was generally willing to take that speculation on himself. He would remark that the formulæ generally given did not exactly represent the conditions of real life. The annuity actually purchaseable, being one payable up to the day of death, was not that of the text books; and the assurance did not become payable at the end of the year, but usually 3 months after proof of the decease of the reversioner. He thought that the amount of assurance required by all the formulæ in connection with reversionary annuities was excessive, and that larger amounts might be raised and the transaction be secured by effecting such an increasing assurance (to increase by a fixed sum annually for, say, 20 years) as would provide for the outlay of premiums and interest, charging the whole by way of ordinary mortgage. He knew that offices had an objection to granting assurances increasing without limit. As showing the expediency of further discussing the subject, he instanced a case which was submitted several years ago to five well-known Actuaries, where the redemption money for the contingent annuity was fixed by A at £282,000, by B at £237,000, by C at £184,000, by D at £154,000, and by E at £47,000. He did not see much in the modifications proposed by Mr. Sprague, since the practical differences between his formulæ and Mr. Jellicoe's were not important; and they did not obviate the necessity of saddling the vendor with the cost of the enormous amount of assurance required in the early years of the transaction.

Mr. Coles, quoting the amounts of the capitals of the existing Reversionary Companies, stated that the total was little more than £2,000,000; and he thought that the amount distributed amongst the Insurance Offices would scarcely be felt. He wished to know whether the reversions offered ran in large amounts, or were of sufficiently uniform amounts to be taken up by Companies of ordinary capital.

Mr. Baden understood Mr. Sprague to treat the question as one, not of profits, but of investment merely. The problem was to fix the rate of interest, so as to secure a return of the capital improved at that rate. He should take the annuity upon a true table of mortality, at 4 per cent interest, if the office generally improved its funds at that rate; and d at 5 per cent. The result would nearly correspond to taking both the annuity and d at 6 per cent, and we should have a better *rationale*.

The PRESIDENT anticipated an extension in this class of business, by reason of the increase in marriage settlements and the improvement in the value of property. Therefore, he did not think that they would be necessarily driving the Reversionary Companies out of the field; at all events, for some considerable time. It was very advisable to see by the mathematical formulæ what terms can be safely given, even if they were higher than those hitherto prevailing. He thought a Company need not actually purchase the annuity, but lay out its money and trust to averages. A higher rate of interest must be allowed for in cases of long deferred reversions, since they were likely to be kept out for a considerable period of the interest upon a heavy investment of capital.

Mr. SPRAGUE, in reply, was glad to have had so full a meeting and that so many gentlemen of practical experience had come forward. By thus interchanging ideas, greater progress was secured than could possibly be by separate study. His experience differed from Mr. Bunyon's in the matter of a return for those assurances given away; as he found that he always sooner or later received a fair equivalent. He certainly agreed with Mr. Bunyon that it was undesirable to spoil the market. He did not propose that there should be no deviation from the rates he named, but desired principally to show how to calculate the real working of the transaction. He contended that the same rate of interest should be maintained throughout the formulæ, since the considerations attending the grant of an annuity to the public differed from those involved in these transactions. The reserved value of the annuity was in fact employed in the transaction itself, and in the others of the same nature, and realized the same rate of interest. He thought the effect of "selection" was traceable for many years, and he therefore objected to the grant of increasing assurances; for without doubt every year the life was, on the average, getting worse. He was aware of the technical imperfection in the formula for the annuity value, as pointed out by Mr. Hodge. But, as the assurance premiums were generally payable yearly, the ordinary correction to add $\frac{1}{4}$ for half yearly payments was a close approximation. To this point he hoped one day to return, if not anticipated. He was afraid that the amounts of reversions offered for sale did not run uniformly. However that was in his opinion no serious objection to Insurance Companies dealing with them. He thought that an office might safely invest 20 per cent of its funds in these securities—for then even the very worst experience would only reduce the average rate of interest realized on the whole assets from, say, 5 per cent to 4 per cent.